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On the K-Functional of Interpolation between L^p and Orlicz Spaces

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P. Nilson and J. Peetre have recently obtained an explicit expression for the interpolation K-functional between the spaces L^p and L^q, $1 \leqslant p, q < \infty$ (see [NP]). In this same paper they also ask for the corresponding extension of their theorem to the case of Orlicz spaces. The present article offers an approach to this situation for a modified K-functional, equivalent to the customary interpolation K-functional. The method we use is an optimization method based on elementary differential calculus.

Hereafter M will denote an Orlicz convex function on $[0, \infty)$ (i.e., a continuous convex increasing function satisfying $M(0) = 0$, $M(1) = 1$, and $M(x) \to \infty$ as $x \to \infty$). We also suppose M has continuous derivative and satisfies the Δ_2-condition (i.e., there exists $K > 0$ so that $M(2t) \leqslant KM(t)$ for all $t > 0$). As usual, the Orlicz space $L^M = L^M[0, \infty)$ is the space of all (equivalence classes of) measurable functions f on $[0, \infty)$ such that

$$\int_0^\infty M\left(\frac{|f(x)|}{\rho}\right) dx < \infty$$

for some $\rho > 0$. The norm in L^M is defined by $\|f\|_M = \inf\{\rho > 0; \int_0^\infty M(|f|/\rho)\, dx \leqslant 1\}$.

L^M is a rearrangement invariant function space (see [LT]) and the integrable simple functions are dense in L^M. In order to ensure the validity

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of our theorems we will suppose the following assumption is satisfied by the function M:

$$\frac{M'(x)}{x^{p-1}} \text{ is a strictly increasing function verifying}$$

$$\lim_{x\to 0}\frac{M'(x)}{x^{p-1}}=0 \qquad \text{and} \qquad \lim_{x\to\infty}\frac{M'(x)}{x^{p-1}}=\infty \qquad (1\leqslant p<\infty). \qquad (*)_p$$

Obviously, condition $(*)_p$ implies

(i) If $f\in L^M$ and $|f|\geqslant 1$, then $f\in L^p$, and

(ii) If $f\in L^p$ and $|f|\leqslant 1$, then $f\in L^M$.

We will consider the space L^p+L^M. For $f\in L^p+L^M$ and $0<t<\infty$ we define the $K_{p,M}$-functional by

$$K_{p,M}(t;f)=\inf\{\rho>0;\, \Phi_{p,M}(t;f/\rho)\leqslant 1\},$$

where

$$\Phi_{p,M}(t;f)=\inf\left(M(\|g\|_p)+\int_0^\infty M(t\,|h|)\,dx\right)$$

and where the inf is extended over all decompositions $f=g+h$ of f, with $g\in L^p$ and $h\in L^M$. If $M(x)=x^q$, $p<q<\infty$, we have

$$K_{p,q}(t;f)=\inf_{\substack{f=g+h\\ g\in L^p,\, h\in L^q}}(\|g\|_p^q+t^q\,\|h\|_q^q)^{1/q}$$

(compare this expression with the one appearing in [HP]).

In any case, the $K_{p,M}$-functional, as a function of f, is a rearrangement invariant norm on L^p+L^M, equivalent to the usual K-functional given by

$$K(t;f)=\inf_{\substack{f=g+h\\ g\in L^p,\, h\in L^M}}(\|g\|_p+t\,\|h\|_M).$$

In this paper we obtain an explicit expression for the $K_{p,M}$-functional when the function M satisfies the natural conditions stated earlier. Our results for the case $M(x)=x^q$, $q>p$, should be compared to those in [NP].

The case $p=1$ is somewhat singular and the expression for $K_{1,M}$ that we obtain is not a particular case of the one we obtain for $K_{p,M}$ when $p>1$ and so we will divide the discussion into two cases.

We refer the reader to [BL, P] for background on interpolation between L^p (and Orlicz) spaces.

INTERPOLATION BETWEEN L^1 AND L^M

We suppose that M satisfies $(*)_1$. The result we get in this situation is the following

THEOREM 1. *Let f be a non-zero function belonging to $L^1 + L^M$ and let f^* be its left-continuous non-increasing rearrangement. If*

$$x_f = \max\left\{x \in [0, \infty);\, M'\left(\int_0^x (f^*(s) - f^*(x))\, ds\right) \leqslant tM'(tf^*(x))\right\}$$

and λ_f is the unique solution of the λ-equation

$$M'\left(\int_0^{x_f} (f^* - \lambda)\, dx\right) = tM'(t\lambda)$$

then

$$\Phi_{1,M}(t; f) = M\left(\int_0^{x_f} (f^* - \lambda_f)\, dx\right) + x_f M(t\lambda_f) + \int_{x_f}^{\infty} M(tf^*)\, dx.$$

The formula we have just stated has a nicer expression when $M(x) = x^q$, $q > 1$. In fact

COROLLARY 1. *If $f \in L^1 + L^q$, $q > 1$,*

$$K_{1,q}(t; f) = \left(\frac{t^q(\int_0^{x_f} f^*)^q}{(x_f + t^{q'})^{q-1}} + t^q \int_{x_f}^{\infty} f^{*q}\right)^{1/q},$$

where

$$x_f = \max\left\{x \in [0, \infty);\, \int_0^x f^* \leqslant f^*(x)(x + t^{q'})\right\}$$

and $1/q + 1/q' = 1$.

Proof of Corollary 1. It is clear that

$$x_f = \max\left\{x \in [0, \infty);\, \int_0^x f^* - xf^*(x) \leqslant t^{q'} f^*(x)\right\}$$

$$= \max\left\{x \in [0, \infty);\, \int_0^x f^* \leqslant f^*(x)(x + t^{q'})\right\}$$

and the λ-equation is now $\int_0^{x_f} f^* - \lambda \cdot x_f = t^{q'}\lambda$.

Then

$$\lambda_f = \frac{1}{x_f + t^{q'}} \int_0^{x_f} f^*,$$

and besides

$$\begin{aligned}\Phi_{1,M}(t; f) &= \left(\int_0^{x_f} f^* - x_f \lambda_f\right)^q + t^q x_f \lambda_f^q + \int_{x_f}^{\infty} t^q f^{*q} \\ &= \left(\int_0^{x_f} f^*\right)^q \left[\left(\frac{t^{q'}}{x_f + t^{q'}}\right)^q + \frac{x_f t^q}{(x_f + t^{q'})^q}\right] + t^q \int_{x_f}^{\infty} f^{*q}.\end{aligned}$$

Thus $K_{1,q}$ has the form stated earlier. ▌

The proof of Theorem 1 is divided into two steps. First we study the case where f is a simple function and then we consider the convergence question.

We begin our study by considering a simple function $s = \sum_{i=1}^n a_i \chi_{A_i}$ with $a_1 \geqslant \cdots \geqslant a_n > 0$, where A_i's are pairwise disjoint and $m(A_i) = m$, $1 \leqslant i \leqslant n$. It is clear that

$$\Phi_{1,M}(t; s) = \inf_g \left(M(\|g\|_1) + \int_0^{\infty} M(t(s-g))\, dx\right),$$

where g belongs to the class of all measurable functions such that $0 \leqslant g \leqslant s$. If $\mathscr{B}$ is the σ-algebra generated by the sets A_i's, $1 \leqslant i \leqslant n$, and $\mathbb{E}^{\mathscr{B}}$ is the conditional expectation operator with respect to $\mathscr{B}$, it is known that $\|\mathbb{E}^{\mathscr{B}} g\|_1 \leqslant \|g\|_1$. Besides, Jensen's inequality shows that

$$\int_0^{\infty} M(t\mathbb{E}^{\mathscr{B}}(s-g))\, dx \leqslant \int_0^{\infty} M(t(s-g))\, dx.$$

Hence

$$\Phi_{1,M}(t; s) = \inf_{\substack{0 \leqslant y_i \leqslant a_i \\ 1 \leqslant i \leqslant n}} M\left(m \sum_1^n y_i\right) + \sum_1^n m M(t(a_i - y_i)).$$

By continuity there exists a point $\bar{y} = (\bar{y}_1 \cdots \bar{y}_n) \in \prod_{1 \leqslant i \leqslant n} [0, a_i]$ so that $\Phi_{1,M}(t; s) = M(m \sum_1^n \bar{y}_i) + \sum_1^n m M(t(a_i - \bar{y}_i))$.

LEMMA 1. *The point $\bar{y}$ may be chosen in such a way that*

(a) *There exist k, $1 \leqslant k \leqslant n$, such that $0 < \bar{y}_i < a_i$ if $1 \leqslant i \leqslant k$ and $\bar{y}_i = 0$ otherwise.*

(b) *The equation* $M'(m\sum_1^k (a_i-\lambda))=tM'(t\lambda)$ *has only one solution* $\lambda_k\in(0, a_k)$ *and besides* $\bar{y}_i=a_i-\lambda_k$ *for* $1\leqslant i\leqslant k$.

(c) $k=\max\{j; M'(m\sum_1^j (a_i-a_j))<tM'(ta_j)\}$.

Proof. Let ψ be the function defined on $\prod_{1\leqslant i\leqslant n}[0, a_i]$ by $\psi(y_1, ..., y_n)=M(m\sum_1^n y_i)+m\sum_1^n M(t(a_i-y_i))$.

Since $1\leqslant i<j\leqslant n$ implies $a_i\geqslant a_j$, the function $h(x)=M(ta_i-tx)-M(ta_j-tx)$ is non-increasing for $0\leqslant x\leqslant a_j$, then

$$\psi(y_1, ..., \overset{i)}{0}, ..., \overset{j)}{y_j}, ..., y_n)\geqslant\psi(y_1, ..., \overset{i)}{y_j}, ..., \overset{j)}{0}, ..., y_n)$$

and so we may suppose $y_{k+1}=\cdots=y_n=0$ for some k. On the other hand, $(\partial\psi/\partial y_1)(0^+, ..., 0)<0$ and $(\partial\psi/\partial y_i)(y_1, ..., \overset{i)}{a_i^-}, ..., y_n)>0$, then $\bar{y}_1>0$ and $\bar{y}_i<a_i$. This completes the proof of part (a).

As a consequence of (a), the point $(\bar{y}_1, ..., \bar{y}_k)$ is the minimum of the function $\psi(y_1, ..., y_k, 0, ..., 0)$. Thus

$$\frac{\partial\psi}{\partial y_i}(\bar{y}_1, ..., \bar{y}_k, 0, ..., 0)=0$$

for all $1\leqslant i\leqslant k$. It implies that

$$M'\left(m\sum_1^k \bar{y}_i\right)=tM'(t(a_i-\bar{y}_i)), \qquad 1\leqslant i\leqslant k,$$

and then we have $a_1-\bar{y}_1=\cdots=a_k-\bar{y}_k=\lambda_k>0$, where λ_k verifies

$$M'\left(m\sum_1^k (a_i-\lambda_k)\right)=tM'(\lambda_k t).$$

This concludes part (b).

Let $\mathscr{A}=\{j; M'(m\sum_1^j (a_i-a_j))<tM'(a_jt)\}$. It is easy to see that 1 and k belong to $\mathscr{A}$, and if $j\in\mathscr{A}$ then $j-1\in\mathscr{A}$ (here we use that M' is strictly increasing). For each $j\in\mathscr{A}$, let λ_j be the corresponding solution of the equation

$$M'\left(m\sum_1^j (a_i-\lambda)\right)=tM'(\lambda t)$$

(this solution necessarily belongs to the open interval $(0, a_j)$). Since $\sum_1^{j-1}(a_i-a_j)=\sum_1^j (a_i-a_j)$ we have

$$M'\left(m\sum_1^{j-1} (a_i-a_j)\right)<tM'(ta_j)$$

so $\lambda_{j-1} < a_j$. Hence, if $j \in \mathscr{A}$ the function $h(x) = \psi(a_1 - \lambda_{j-1}, ..., a_{j-1} - \lambda_{j-1}, x, 0 \cdots 0)$ verifies $h'(0^+) < 0$ and then $j-1 \neq k$. It shows that $k = \max \mathscr{A}$ and we get part (c). ∎

The preceding lemma enables us to prove the theorem for suitable simple functions. We state this result in the following

PROPOSITION 1. *If $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ is a simple function as before,*

$$\Phi_{1,m}(s) = M\left(m \sum_{1}^{k} (a_i - \lambda_k)\right) + kmM(t\lambda_k) + \sum_{k+1}^{n} mM(ta_i),$$

where $k = \max\{j; M'(m \sum_1^j (a_i - a_j)) \leqslant tM'(ta_j)\}$ and λ_k is the unique solution of the equation

$$M'\left(m \sum_{1}^{k} (a_i - \lambda)\right) = tM'(t\lambda).$$

Proof. Let $\bar{k} = \max\{j; M'(m \sum_1^j (a_i - a_j)) \leqslant tM'(a_j)\}$. If $k = \max\{j; M'(m \sum_1^j (a_i - a_j)) < tM'(ta_j)\}$ it is clear that $k \leqslant \bar{k}$. If $k < \bar{k}$ we only need to prove that $\lambda_k = \lambda_{\bar{k}}$ and

$$M\left(m \sum_{1}^{k} (a_i - \lambda_k)\right) + kmM(t\lambda_k) + \sum_{k+1}^{\bar{k}} mM(ta_i)$$

$$= M\left(m \sum_{1}^{k} (a_i - \lambda_{\bar{k}})\right) + kmM(t\lambda_{\bar{k}}).$$

For $k+1 \leqslant j \leqslant \bar{k}$ we have

$$M'\left(m \sum_{1}^{j} (a_i - a_j)\right) = tM'(ta_j)$$

then $\lambda_j = a_j$. Since $\sum_1^j (a_i - a_j) = \sum_1^{j-1} (a_i - a_j)$ we obtain $a_j = \lambda_{j-1}$ and so $\lambda_{\bar{k}} = \cdots = \lambda_k = a_{\bar{k}} = \cdots = a_{k+1}$. ∎

Proof of Theorem 1. Without loss of generality we assume $f = f^*$ (this simplifies the notation). We define the function F by

$$F(x) = M'\left(\int_0^x (f(s) - f(x)\, ds) - tM'(tf(x))\right), \qquad x > 0.$$

This function F has the following properties

LEMMA 2. (a) *F is left-continous for $x>0$ and there exists $F(0)=\lim_{x\to 0^+} F(x)<0$.*

(b) *F is non-decreasing.*

(c) *The set $\{x\in[0,\infty); F(x)\leqslant 0\}$ is a compact subinterval; we will denote by x_f its maximum.*

(d) *There exists a unique solution of the equation*

$$M'\left(\int_0^{x_f}(f-\lambda)\,dx\right)=tM'(t\lambda).$$

This solution denoted by λ_f is in the interval $(0, x_f]$.

(e) *For the simple functions appearing in the preceding proposition $x_s=km$ and $\lambda_s=\lambda_k$.*

Proof. Assertion (a) is a conseqence of the dominated convergence theorem. Part (b) can be deduced from the fact that if $x'<x$ then

$$\int_0^x (f-f(x))\,ds\geqslant\int_0^{x'}(f-f(x'))\,dx.$$

In order to show (c) we note that $\lim_{x\to\infty} f(x)=0$. If the interval $\{x; F(x)\leqslant 0\}$ were not bounded, we would have

$$M'\left(\int_0^1 (f-f(x))\,ds\right)\leqslant M'\left(\int_0^x (f-f(x))\,ds\right)$$
$$\leqslant tM'(tf(x)),$$

whenever $x>1$. Thus $M'(\int_0^1 f)=0$, which would imply $f=0$.

Since M' is strictly increasing we have (d). Part (e) can be easily computed. ∎

Next we go on with the proof of Theorem 1. For each $n\in\mathbb{N}$, let s_n be the simple function defined by

$$s_n=f\left(\frac{1}{2^n}\right)\chi_{[0,1/2^n]}+\sum_{i=2}^{n2^n} f\left(\frac{i}{2^n}\right)\chi_{((i-1)/2^n,\,i/2^n]}.$$

It is obvious that $s_n\nearrow f$ almost elverywhere when n goes to ∞ (more precisely, $\lim_{n\to\infty} s_n(x)=f(x)$ except at most in the discontinuity points of f). Now we shall show that $\lim_n x_{s_n}=x_f$ and $\lim_n \lambda_{s_n}=\lambda_f$. Given $\varepsilon>0$ we can choose natural numbers k_0, n_0 in such a way that $a=k_0/2^{n_0}<n_0 2^{n_0}$ and $x_f-\varepsilon<a<x_f$.

Since we may express $a=k_n/2^n$ for every $n \geqslant n_0$ ($k_n \in \mathbb{N}$) and $f(a)=s_n(a)$, we have

$$M'\left(\int_0^a (s_n - s_n(a))\,dx\right) \leqslant M'\left(\int_0^a (f-f(a))\,dx\right)$$
$$\leqslant tM'(tf(a)) = tM'(ts_n(a))$$

for every $n \geqslant n_0$. Hence $x_f - \varepsilon < a \leqslant x_{s_n}$ if $n \geqslant n_0$. On the other hand, let k_1, n_1 be natural numbers so that $x_f < b = k_1/2^{n_1} < x_f + \varepsilon$. The same reasons as before imply $s_n(b) = f(b)$. Thus

$$M'\left(\int_0^b (f-f(b))\,dx\right) > tM'(tf(b)) = tM'(ts_n(b)).$$

Since $s_n \nearrow f$ as $n \to \infty$, the monotone convergence theorem shows that

$$\int_0^b (s_n - f(b))\,dx \underset{n}{\longrightarrow} \int_0^b (f-f(b))\,dx.$$

Then for n large enough

$$M'\left(\int_0^b (s_n - s_n(b))\,dx\right) > tM'(ts_n(b))$$

and it concludes the proof of $\lim_n x_{s_n} = x_f$.

Now consider the functions

$$H_n(\lambda) = M'\left(\int_0^{x_{s_n}} (s_n - \lambda)\,dx\right) - tM'(t\lambda),$$

$n \in \mathbb{N}$, and

$$H_f(\lambda) = M'\left(\int_0^{x_f} (f-\lambda)\,dx\right) - tM'(t\lambda).$$

We recall that the sequence $(\lambda_{s_n})_n$ is bounded ($\lambda_{s_n} \leqslant s_n(x_{s_n}) = f(x_{s_n})$). If $\bar{\lambda}$ is a limit point of a convergent subsequence $(\lambda_{s_{n'}})_{n'}$ of (λ_{s_n}), we have

$$|H_{n'}(\lambda_{s_{n'}}) - H_f(\bar{\lambda})| \leqslant |H_{n'}(\lambda_{s_{n'}}) - H_f(\lambda_{s_{n'}})|$$
$$+ |H_f(\lambda_{s_{n'}}) - H_f(\bar{\lambda})| \underset{n \to \infty}{\longrightarrow} 0$$

since $H_n \to_n H_f$ uniformly on bounded intervals and $\lim_n \int_0^{x_{s_n}} s_n = \int_0^{x_f} f$. Hence $H_f(\bar{\lambda}) = 0$, which implies $\bar{\lambda} = \lambda_f = \lim_n \lambda_{s_n}$.

By combining these results and applying the monotone and dominated convergence theorems we have shown that

$$\lim_n \Phi_{1,M}(s_n) = M\left(\int_0^\infty (f-h)\,dx\right) + \int_0^\infty M(th)\,dx,$$

where $h=\lambda_f\chi_{[0,x_f]}+f\chi_{[x_f,\infty)}\in L^M$. Thus $\Phi_{1,M}(f)\leqslant \lim_n \Phi_{1,}(s_n)$. And we end the proof by recalling that $\Phi_{1,M}(s_n)\leqslant \Phi_{1,M}(f)$. ∎

Remark. If we take for granted that the extremal decomposition is obtained by horizontal slicing of the function we get our result in a much simpler manner.

INTERPOLATION BETWEEN L^p AND L^M, $p>1$

Now we assume that M satisfies $(*)_p$. In this case we will follow the same procedure as before, although surprisingly the results we will obtain are formally different. We are not able to obtain more explicit expression, even for $M=t^q$, of the critical points which minimize the corresponding infimum, but we are sure that the optimal decomposition does not correspond to a horizontal slicing of the function. The theorem we shall prove can be stated in the following way

THEOREM 2. *If $f\in L^p+L^M$, $f\neq 0$, there exists a unique non-increasing non-negative function $g\in L^p$ such that*

(i) $f^*-g\in L^M$, $0<g(x)<f^*(x)$, *and*

$$\frac{M'(\|g\|_p)}{\|g\|_p^{p-1}} = t\,\frac{M'(t(f^*(x)-g(x)))}{g(x)^{p-1}}$$

for almost all $x\in \operatorname{supp} f^*$,

(ii) $\Phi_{p,M}(t;f)=M(\|g\|_p)+\int_0^\infty M(t(f^*-g))\,dx$ *(f^* is the left-continuous non-increasing rearrangement of f).*

For $M(x)=x^q$ $(q>p)$ we obtain

COROLLARY 2. *If $f\in L^p+L^q$, $1<p<q<\infty$, there exists a unique non-increasing non-negative function $g\in L^p$ verifying*

(i) $f-g\in L^q$, $0<g(x)<f^*(x)$, *and*

$$\|g\|_p^{q-p} = t^q\,\frac{(f^*(x)-g(x))^{q-1}}{g(x)^{p-1}}$$

almost everywhere on $\operatorname{supp} f^*$;

(ii) $K_{p,q}(t;f)=t(\int_0^\infty f^*(f^*-g)^{q-1})^{1/q}$.

Proof of Corollary 2. The expression appearing in (i) is clear. In order to show (ii) we note that

$$\int_0^\infty (f^*-g)^{q-1}g = t^{-q}\,\|g\|_p^{q-p}\int_0^\infty g^p = t^{-q}\,\|g\|_p^q.$$

Thus

$$\Phi_{p,q}(t;f) = \|g\|_p^q + t^q\int_0^\infty f^*(f^*-g)^{q-1} - \|g\|_p^q$$

$$= t^q\int_0^\infty f^*(f^*-g)^{q-1}.$$

Proof of Theorem 2. We begin by considering simple functions in the same manner as before. If $s=\sum_1^n a_i\chi_{A_i}$, where $a_1\geqslant\cdots\geqslant a_n>0$, A_i's are pairwise disjoint with $m(A_i)=m$, $1\leqslant i\leqslant n$. It is again clear that

$$\Phi_{p,M}(s) = \inf_{\substack{0\leqslant y_i\leqslant a_i\\ 1\leqslant i\leqslant n}} \psi(y_1,\ldots,y_n),$$

where

$$\psi(y_1,\ldots,y_n) = M\left(\left(\sum_1^n m y_i^p\right)^{1/p}\right) + \sum_1^n mM(t(a_i-y_i)).$$

Since $(\partial\psi/\partial y_i)(y_1,\ldots,\overset{i)}{0^+},\ldots,y_n)<0$ and $(\partial\psi/\partial y_i)(y_1,\ldots,\overset{i)}{a_i^-},\ldots,y_n)>0$, $1\leqslant i\leqslant n$, the minimum of ψ is attained at an interior point of the compact $\prod_1^n[0,a_i]$. If $\bar y=(\bar y_1,\ldots,\bar y_n)$ is such a point it verifies

$$\frac{M'((\sum_1^n m\bar y_i^p)^{1/p})}{(\sum_1^n m\bar y_i^p)^{1-1/p}} = t\,\frac{M'(t(a_i-\bar y_i))}{\bar y_i^{p-1}}, \qquad 1\leqslant i\leqslant n. \tag{1}$$

Now we shall prove that

(i) $\bar y_1\geqslant\cdots\geqslant\bar y_n$.

(ii) The system (1) has only one solution.

For each $a>0$ we consider the strictly decreasing function h_a defined by

$$h_a(y) = t\,\frac{M'(t(a-y))}{y^{p-1}}, \qquad 0<y\leqslant a.$$

If $a>a'>0$ and $0<y<a'$ we have $h_a(y)>h_{a'}(y)$, so (i) is easily verified. Now suppose $\bar{\bar y}=(\bar{\bar y}_1,\ldots,\bar{\bar y}_n)$ is another solution of (1). If $\|\bar y\|_p<\|\bar{\bar y}\|_p$ then $h_{a_i}(\bar y_i)<h_{a_i}(\bar{\bar y}_i)$, $1\leqslant i\leqslant n$, because of the condition $(*)_p$ imposed on M.

This implies $\bar{y}_i > \bar{\bar{y}}_i$, $1 \leqslant i \leqslant n$, which contradicts the assumption that $\|\bar{y}\|_p < \|\bar{\bar{y}}\|_p$. Hence $\|\bar{y}\|_p = \|\bar{\bar{y}}\|_p$ and so $\bar{y}_i = \bar{\bar{y}}_i$, for all $1 \leqslant i \leqslant n$. Consequently (ii) is proved and we have just established the following

PROPOSITION 2. *Let $s = \sum_1^n a_i \chi_{A_i}$, $a_1 \geqslant \cdots \geqslant a_n > 0$, $m(A_i) = m$, $1 \leqslant i \leqslant n$, and the A_i's pairwise disjoint. There exists a unique function $g \in L^p$ such that*

$$\Phi_{p,M}(s) = M(\|g\|_p) + \int_0^\infty M(t(s-g))\, dx.$$

Moreover, if $x \in \operatorname{supp} s$, $0 < g(x) < s(x)$ and

$$\frac{M'(\|g\|_p)}{\|g\|_p^{p-1}} = \frac{tM'(t(s(x)-g(x)))}{g(x)^{p-1}}.$$

(Obviously $g = \sum_1^n \bar{y}_i \chi_{A_i}$, where $(\bar{y}_i)_1^n$ is the solution of the system (1).)

Suppose now f is a non-increasing non-negative function in $L^p + L^M$. By using the same arguments as in the proof of Theorem 1, there exists a sequence of simple functions $(s_n)_n$ such that $0 \leqslant s_n \nearrow f$ [a.e.]. For each $n \in \mathbb{N}$ let g_n be the corresponding function associated to s_n according to the preceding proposition. For each $a > 0$, consider again the function $h_a(y) = tM'(t(a-y))/y^{p-1}$, $0 < y \leqslant a$. Since $h_{s_n(x)} \leqslant h_{s_{n+1}(x)}$,

$$h_{s_n(x)}(g_n(x)) = \frac{M'(\|g_n\|_p)}{\|g_n\|_p^{p-1}} \qquad \text{[a.e.]}$$

and the same for s_{n+1}, we have that $\|g_n\|_p > \|g_{n+1}\|_p$ would imply $g_n(x) < g_{n+1}(x)$ [a.e.], which is not possible, so the sequence $(\|g_n\|_p)_n$ is non-decreasing. As also $M(\|g_n\|_p) \leqslant \Phi_{p,M}(t; s_n) \leqslant \Phi_{p,M}(t, f)$ we know that there exists $C = \lim_n (M(\|g_n\|_p)/\|g_n\|_p^{p-1}) < \infty$. Let x be such that $f(x) > 0$. The equation $h_{f(x)}(y) = C$ has only one solution denoted by $g(x)$ and belonging to the open interval $(0, f(x))$. We will show that $\lim_n g_n(x) = g(x)$ [a.e.], $x \in \operatorname{supp} f$. Suppose that $f(x) = \lim_n s_n(x)$ and fix $\varepsilon > 0$. We can choose k, $0 < k < g(x) - \varepsilon$, satisfying $h_{f(x)}(y) < 2g(x)$ whenever $y \geqslant k$. Denoting $\delta = \min\{h_{f(x)}(g(x)-\varepsilon) - C,\ C - h_{f(x)}(g(x)+\varepsilon)\}$, since M' is uniformly continuous on $[0, t(f(x)-k)]$, we have

$$|h_{f(x)}(y) - h_{f_n(x)}(y)| < \delta$$

if $y \in [k, f_n(x)]$ and n is large enough. Thus

$$h_{f_n(x)}^{-1}([C-\delta, C]) \subseteq (g(x)-\varepsilon,\ g(x)+\varepsilon)$$

and hence $g_n(x) \in (g(x)-\varepsilon,\ g(x)+\varepsilon)$ for $n \geqslant n_0$.

The function $g \in L^p$ and $\|g\|_p \leqslant \lim_n \|g_n\|_p < \infty$ since the g_n's are obviously non-increasing. Fatou's lemma shows that

$$\int_0^\infty M(t(f-g)) \leqslant \underline{\lim}_n \int_0^\infty M(t(s_n-g))$$
$$= \lim \Phi_{p,M}(t, s_n) - \lim M(\|g_n\|_p)$$
$$\leqslant \Phi_{p,M}(t; f) - M(\|g\|_p).$$

Then $f-g \in L^M$ and eventually

$$\Phi_{p,M}(t; f) = M(\|g\|_p) + \int_0^\infty M(t(f-g)),$$

which concludes the proof of the theorem. ∎

Remark. Corollary 1 has to be compared with the results of [NP], where the usual K-functional is computed.

In this case the extremal decomposition is also obtained by horizontal slicing of the function, but the cutting level is given by a different equation, namely (when f^* is continuous)

$$xf^*(x)^q + \int_x^\infty f^{*q} = f^*(x)^q t^{q'}.$$

If $= g + h$ is the extremal decomposition, Nilson and Peetre's result states that $t^{q'} \|h\|_\infty = t\, \|h\|_q$, while ours says that $t^{q'}\, \|h\|_\infty = \|g\|_1$.

The result of [NP] can easily be translated to the Orlicz space case $[L^1, L^M]$. In this setting the relevant norm on L^M is the "Orlicz norm," not the ordinary one ("Luxemburg norm"), i.e., the dual norm of the Luxemburg norm associated to the Young conjugate function M^* (see [LT]). Then

$$K(t; f) = \inf_{g+h=f} (\|g\|_1 + t\, \|h\|_M)$$

is given by the formula

$$K(t; f) = \int_0^x f(u)\, du + \int_x^\infty f^*(u)\, \Psi\left(\frac{\alpha f^*(u)}{f^*(x)}\right) du,$$

where $\Psi(\lambda) = M^{*\prime -1}(\lambda)$ is the reciprocal of the derivative of M^*, α is a normalization factor ($\Psi(\alpha) = 1$), and x is defined by the equation

$$1 = xM^*\left(\frac{1}{t}\right) + \int_x^\infty M^*\left[\frac{1}{t}\, \Psi\left(\frac{\alpha f^*(u)}{f^*(x)}\right)\right] du$$

(when f^* is continuous).

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